Fig. 6.7. Experiment of the Innsbruck group. The pair of entangled photons is produced in a nonlinear crystal. After A. Zeilinger, Rev. Mod. Phys. 71, S288 (1999).

both travel to the right with a small variable angle between their trajectories (Fig. 6.7). In order to obtain the trajectory of photon 1, it is sufficient to reverse its direction of propagation when leaving the plate in Figs. 6.5 and 6.6. The experiment confirms the preceding discussion in all respects (Fig. 6.8).

6.3.4 Three-particle entangled states (GHZ states)

GHZ (Greenberger–Horne–Zeilinger) states are three-particle entangled states which exhibit nonclassical properties in an even more spectacular fashion than two-particle states. It is known how to create three-photon entangled states experimentally using parametric conversion. To simplify the discussion, we shall limit ourselves to the theory of entangled states of three spin-1/2 particles. We assume that an unstable particle decays

Fig. 6.8. Interference observed by the Innsbruck group. After A. Zeilinger, Rev. Mod. Phys. 71, S288 (1999).
into three identical particles of spin $1/2$ which are emitted in a plane in a configuration in which the three momenta lie at angles of $2\pi/3$ to each other, and the three particles are in the entangled spin state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |+++)\rangle - |--\rangle\right).$$  \hfill (6.54)

Three experimentalists, Alice ($a$), Bob ($b$), and Charlotte ($c$), can measure the spin component in the direction perpendicular to the direction of propagation of each particle (Fig. 6.9). The momenta lie in the horizontal plane, and the $Oz$ axis is chosen to lie along the propagation direction (so that it depends on the particle), while $Oy$ is vertical and $\hat{x} = \hat{y} \times \hat{z}$. Let us examine the three following operators:

$$\Sigma_a = \sigma_{ax} \sigma_{by} \sigma_{cx}, \quad \Sigma_b = \sigma_{ay} \sigma_{bx} \sigma_{cy}, \quad \Sigma_c = \sigma_{az} \sigma_{by} \sigma_{cx}.$$  \hfill (6.55)

The matrices $\sigma_i$ act in the space of spin states of particle $i$, $i = a, b, c$. The index $i$ of $\Sigma_i$ specifies the position of the matrix $\sigma_x$ in the products (6.55). The three operators $\Sigma_i$ commute with each other. To show this, we use the fact that $\sigma$ matrices acting on different spaces commute, for example

$$\sigma_{ax} \sigma_{by} = \sigma_{by} \sigma_{ax}.$$  

For matrices acting in the same space we use (3.48):

$$\sigma_x \sigma_y = -\sigma_y \sigma_x,$$

as well as

$$\sigma_x^2 = \sigma_y^2 = I.$$

![Fig. 6.9. Configuration of a GHZ type of experiment.](image-url)
As an example, let us show that $\Sigma_a$ and $\Sigma_b$ commute owing to the fact that the two operators $\Sigma_a \Sigma_b$ and $\Sigma_b \Sigma_a$ differ by an even number of anticommutations:

$$\Sigma_a \Sigma_b = \sigma_{\alpha a} \sigma_{\alpha b} \sigma_{\beta b} \sigma_{\beta a} \sigma_{\gamma b} \sigma_{\gamma a} = \sigma_{\alpha a} \sigma_{\beta b} \sigma_{\gamma a} \sigma_{\gamma b} = \sigma_{\alpha a} \sigma_{\beta b} \sigma_{\gamma a} \sigma_{\gamma b} = \Sigma_b \Sigma_a.$$

The other commutation relations are demonstrated in a similar fashion. The squares of the operators $\Sigma_i$ are unit operators ($\Sigma_i^2 = 1$), their eigenvalues are $\pm 1$, and, as they commute with each other, they can be simultaneously diagonalized. There then exists an eigenvector $|\Psi\rangle$ preserving the symmetry between the three particles constructed explicitly in (6.54) such that

$$\Sigma_a |\Psi\rangle = \Sigma_b |\Psi\rangle = \Sigma_c |\Psi\rangle = |\Psi\rangle.$$  \hspace{1cm} (6.56)

Equation (6.56) can be shown directly by examining the action of $\Sigma_i$ on $|\Psi\rangle$ using the following properties

$$\sigma_x |+\rangle = |-\rangle, \quad \sigma_x |-\rangle = |+\rangle,$$

$$\sigma_y |+\rangle = i|-\rangle, \quad \sigma_y |-\rangle = -i|+\rangle.$$

The spins are measured in the configurations $(x, y, y)$, $(y, x, y)$, and $(y, y, x)$. For example, in the configuration $(x, y, y)$, Alice orients her Stern–Gerlach apparatus in the direction $Oy$, and Bob and Charlotte orient theirs in the direction $Ox$. Measurements of $\sigma_x$ or of $\sigma_y$ always give the result $\pm 1$, and if the particle triplet is in the state $|\Psi\rangle$, the product of the results of Alice, Bob, and Charlotte will be $+1$ for any configuration of measurement devices.

Let us now turn to the configuration $(x, x, x)$ by examining the action of the operator

$$\Sigma = \sigma_{\alpha x} \sigma_{\beta x} \sigma_{\gamma x}$$

on $|\Psi\rangle$. The product of the results of spin measurements in the configuration $(x, x, x)$ will always be $-1$ because

$$\Sigma |\Psi\rangle = -|\Psi\rangle,$$  \hspace{1cm} (6.57)

as is easily checked by allowing $\sigma_{\alpha x} \sigma_{\beta x} \sigma_{\gamma x}$ to act on $|\Psi\rangle$:

$$\sigma_{\alpha x} \sigma_{\beta x} \sigma_{\gamma x} |\Psi\rangle = \sigma_{\alpha x} \sigma_{\beta x} \sigma_{\gamma x} \left( \frac{1}{\sqrt{2}} \left( |++-\rangle - |--\rangle \right) \right)$$

$$= \frac{1}{\sqrt{2}} \left( |--\rangle - |++-\rangle \right) = -|\Psi\rangle.$$

Let us now confront the above results with local realism. Once the three particles are sufficiently far apart, each of them possesses its own physical characteristics. We use $A_x$ to denote the result of measuring the $x$ component of the spin of particle $a$ by Alice, ..., $C_y$ the result of measuring the $y$ component of the spin of particle $c$ by Charlotte, and
so on, with \( A_1, \ldots, C_y = \pm 1 \). When the \( x \) component is measured in conjunction with two measurements of the \( y \) component, we have seen (see (6.56)) that the product of the results is \(+1\):

\[
A_x B_y C_y = +1, \quad A_x B_x C_y = +1, \quad A_y B_x C_y = +1. \quad (6.58)
\]

However, when the particles are in flight, two of the three experimentalists can decide to modify the direction of their analyzer axes, orienting them in the \( Ox \) direction. Then the product of the three spin components will be \(-1\):

\[
A_x B_x C_x = -1. \quad (6.59)
\]

However, we note that

\[
A_x B_x C_x = (A_x B_x C_y)(A_x B_y C_y)(A_x B_y C_x) = 1
\]

because \( A_y^2 = B_y^2 = C_y^2 = 1 \). Equations (6.58) and (6.59) are incompatible. We do not have an inequality based on statistical correlations as in Section 6.3.2, but instead a perfect anticorrelation! Local realism would mean that the property \( \sigma_{ux} \) has a physical reality in the EPR sense, since it can be measured without disturbing particle \( a \) by measuring \( \sigma_{xy} \), and \( \sigma_{yx} \): \( A_x = B_y C_y \). However, it is also possible to obtain \( A_x \) by measuring \( \sigma_{yx} \) and \( \sigma_{zy} \): \( A_x = -B_x C_y \). Local realism implies that it is the same \( A_x \), but this is not the case in quantum mechanics. The value of \( A_x \) is contextual; it depends on physical properties incompatible with each other which are measured simultaneously with \( \sigma_{ux} \), and \( A_x \) in (6.58) is not the same as \( A_x \) in (6.59). As in the case of the Bell inequalities, the problem arises because it is not possible to simultaneously measure the six quantities \( A_x, \ldots, C_y \), which are the eigenvalues of operators which do not all commute with each other, and the simultaneous measurement of these six quantities is counterfactual: at most three can be measured in a given experiment. The operators \( \Sigma_a, \Sigma_b, \Sigma_c \), and \( \Sigma \) all commute with each other, because \( \Sigma \) is a function of the commuting operators \( \Sigma_a, \Sigma_b, \Sigma_c \):

\[
\Sigma = -\Sigma_a \Sigma_b \Sigma_c.
\]

It is therefore possible to imagine an experiment where they are all four measured simultaneously. Such an experiment could not be performed by measuring the spins separately, and as in the case of teleportation (Section 6.4.2), it would be necessary to use an interaction between the spins. However, local realism also requires that measurement of the product \( \Sigma_a \Sigma_b \Sigma_c \) gives a result identical to the product of the individual values of the spin operators, which is a statement incompatible with quantum physics.

6.4 Applications

6.4.1 Measurement and decoherence

In the Bohr or Copenhagen interpretation – or rather noninterpretation; see A. Leggett in Further Reading – of measurement in quantum mechanics, the measuring device operates according to macroscopic laws: the result of the measurement is read, for